# Math Logic: Model Theory & Computability
## Lecture 26

### Computable functions and relations.

There are many definitions of computability and all of them are equivalent, which suggests that this is the "correct" notion capturing our intuitive understanding of what algorithm is.

**Def.** For $k \in \mathbb{N}$, a relation $R \subseteq \mathbb{N}^{k+1}$ and $\vec{a} \in \mathbb{N}^k$, we let $\mu_x(R(\vec{a}, x))$ denote the least $x \in \mathbb{N}$ such that $R(\vec{a}, x)$ holds if such an $x$ exists; otherwise we say that $\mu_x(R(\vec{a}, x))$ is undefined, and write $\mu_x(R(\vec{a}, x)) = \perp$. This is called the search or minimization operation.

**Def.** A function $f: \mathbb{N}^k \to \mathbb{N}$ is called computable if it is either one of the functions in (C1) or is obtained from the functions in (C1) by finitely many applications of the operations (C2) and (C3).

(C1) Primitives:

(i) addition $+ : \mathbb{N}^2 \to \mathbb{N}$ by $(x, y) \mapsto x + y$.

(ii) multiplication: $\cdot : \mathbb{N}^2 \to \mathbb{N}$ by $(x, y) \mapsto x \cdot y$.

(iii) less than or equal to: $\mathbb{1}_{\leq} : \mathbb{N}^2 \to \mathbb{N}$ by $(x, y) \mapsto \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise.} \end{cases}$

(iv$^\infty$) projections: for each $i \leq k$, the function $P_i^k : \mathbb{N}^k \to \mathbb{N}$ by $(x_1, x_2, \ldots, x_k) \mapsto x_i$.

(C2) Composition: if $g: \mathbb{N}^m \to \mathbb{N}$ and $h_i : \mathbb{N}^k \to \mathbb{N}$, for $i = 1, \ldots, m$, are computable then so is $f: \mathbb{N}^k \to \mathbb{N}$ by $\vec{a} \mapsto g(h_1(\vec{a}), \ldots, h_m(\vec{a}))$.

(C3) Successful search: if $g : \mathbb{N}^{k+1} \to \mathbb{N}$ is computable and is such that for each $\vec{a} \in \mathbb{N}^k$ there is $x \in \mathbb{N}$ with $g(\vec{a}, x) = 0$, then the function $f : \mathbb{N}^k \to \mathbb{N}$ by $\vec{a} \mapsto \mu_x (g(\vec{a}, x) = 0)$ is computable. In this case we say that $f$ is obtained from $g$ by successful search.

We say that a relation $R \subseteq \mathbb{N}^k$ is computable if such is its indicator function $\mathbb{1}_R : \mathbb{N}^k \to \mathbb{N}$, defined by $\vec{a} \mapsto \begin{cases} 1 & \text{if } R(\vec{a}) \text{ holds} \\ 0 & \text{otherwise.} \end{cases}$
For example, $\leq$ is computable by (C1)(iii).

Prop. All computable functions and relations are arithmetical, i.e. definable in $\underline{N} := (\mathbb{N}, 0, S, +, \cdot)$.

Proof. (C1)(i) and (ii) are definable by definition. (C1)(iii) is definable by the formula $\varphi_{\leq}(x,y,z) := (\leq(x,y) \to z=1) \wedge (\neg \leq(x,y) \to z=0)$, where $\leq(x,y) := \exists u\, (x+u=y)$. Finally (C1)(iv) for $i \leq k$ is definable by $\Pi_i^k(x_1, x_2, \dots, x_k, y) := (x_i = y)$.

That definable functions are closed under compositions was shown in HW. As for successful search, if $f(\vec{a}) := \mu_x (g(\vec{a}, x) = 0)$ and $g$ is arithmetical, i.e. definable by a formula $\varphi_g(\vec{y}, x, u)$ then $f$ is definable by the formula

$$\varphi_f(\vec{y}, x) := \varphi_g(\vec{y}, x, 0) \wedge \forall v\, (v < x \to \neg \varphi_g(\vec{y}, v, 0)),$$

where $v < x$ stands for $\exists t\, (t \neq 0 \wedge v + t = x)$. $\square$

We have chosen a minimalistic definition of computability so that the previous proposition would be easy to prove. We will now work towards showing that the class of computable functions is rich and in particular, is closed under the following operation.

(C4) **Primitive recursion:** For $k \in \mathbb{N}$ and functions $g: \mathbb{N}^k \to \mathbb{N}$, $h: \mathbb{N}^{k+2} \to \mathbb{N}$, we say that the function $f: \mathbb{N}^{k+1} \to \mathbb{N}$ is obtained from $g, h$ by **primitive recursion** if for each $\vec{a} \in \mathbb{N}^k$ and $n \in \mathbb{N}$, we have:

$$\begin{cases} f(\vec{a}, 0) = g(\vec{a}) \\ f(\vec{a}, n+1) = h(\vec{a}, n, f(\vec{a}, n)) \end{cases}.$$

We will spend the next two lectures proving the following:

**Theorem.** The class of computable functions is closed under primitive recursion, i.e. if $g, h$ as above computable and $f$ is obtained from $g, h$ by primitive recursion, then $f$ is computable.

**Example.** The function $f_2: \mathbb{N} \to \mathbb{N}$, $n \to 2^n$ is obtained from the constant function $1$ and multiplication by $2$ primitive recursion:

$$\begin{cases} 2^0 = 1 = g() \\ 2^{n+1} = 2 \cdot 2^n = h(n, 2^n) \end{cases}.$$

more precisely, here $g: \underset{\{\emptyset\}}{\overset{\shortparallel}{\mathbb{N}^0}} \to \mathbb{N}$ by $\emptyset \mapsto 1$ and $h: \mathbb{N}^2 \to \mathbb{N}$, $(n, y) \mapsto 2 \cdot y$.

We will quickly show below that $g$ and $h$ are computable, so the theorem above will give us that $n \to 2^n$ is also computable.

**Prop.**

(a) The relations $\geq$ and $=$ are computable.

*Proof.* Note that $x \geq y$ iff $y \leq x$, and to swap the inputs we use projections: $\mathbb{1}_{\geq}(x_1, x_2) = \mathbb{1}_{\leq}(x_2, x_1) = \mathbb{1}_{\leq}(P^2_2(x_1, x_2), P^2_1(x_1, x_2))$.

Lastly, note that $\mathbb{1}_{=}(x_1, x_2) = \mathbb{1}_{\leq}(x_1, x_2) \cdot \mathbb{1}_{\geq}(x_1, x_2)$. $\boxed{a}$

(b) Constant functions are computable, i.e., for each $m, k \in \mathbb{N}$, the function $C^k_m : \mathbb{N}^k \to \mathbb{N}$, given by $\vec{a} \mapsto m$, is computable.

Proof. We prove by induction on $m$ that for all $k \in \mathbb{N}$, $C^k_m$ is computable.
Base: $m = 0$. $C^k_0(\vec{a}) = \mu_x\left(P^{k+1}_{k+1}(\vec{a}, x) = 0\right)$, so $g = P^{k+1}_{k+1}$.

Step: $m \Rightarrow m+1$. Suppose $C^\ell_m : \mathbb{N}^\ell \to \mathbb{N}$ is computable for all $\ell \in \mathbb{N}$.
Note that $x < y$ iff $x \not\geq y$ iff $\mathbb{1}_{\geq}(x, y) = 0$. Thus,

$$C^k_{m+1}(\vec{a}) = \mu_x\left(C^k_m(\vec{a}) < x\right) = \mu_x\left(\mathbb{1}_{\geq}\left(C^k_m(\vec{a}), x\right) = 0\right) =$$

$$= \mu_x\left(\mathbb{1}_{\geq}\left(C^{k+1}_m(\vec{a}, x), P^{k+1}_{k+1}(\vec{a}, x)\right) = 0\right).$$

$\boxed{b}$

(c) The successor function $S : \mathbb{N} \to \mathbb{N}$ by $n \mapsto n+1$ is computable.
Proof. $S(x) = x + 1 = x + C^1_1(x) = P^1_1(x) + C^1_1(x)$.

$\boxed{c}$

(d) The set of computable relations is a Boolean algebra, i.e. is closed under complements and finite intersections (hence also finite unions).

Proof. If $R_1, R_2 \subseteq \mathbb{N}^k$ are computable $k$-ary relations, i.e. $\mathbb{1}_{R_1}$ and $\mathbb{1}_{R_2}$ are computable functions, then the indicator function $\mathbb{1}_{R_1 \cap R_2}(\vec{a}) = \mathbb{1}_{R_1}(\vec{a}) \cdot \mathbb{1}_{R_2}(\vec{a})$ is computable.

If $R \subseteq \mathbb{N}^k$ is a computable relation, i.e. $\mathbb{1}_R$ is computable, then

$$\begin{aligned} R(\vec{a}) \text{ fails} \quad &\text{iff} \quad \mathbb{1}_R(\vec{a}) = 0 \\ &\text{iff} \quad \mathbb{1}_R(\vec{a}) = C^k_0(\vec{a}), \end{aligned}$$

so $\mathbb{N}^k \setminus R$ is computable by (a).

$\boxed{d}$

(e) Computable functions are closed under successful search applied to any computable relation, i.e. if $R \subseteq \mathbb{N}^{k+1}$ is a computable relation such that for each $\vec{a} \in \mathbb{N}^k$ there is $x \in \mathbb{N}$ such that $R(\vec{a}, x)$ holds, then the function $f: \mathbb{N}^k \to \mathbb{N}$ given by $\vec{a} \mapsto \mu_x(R(\vec{a}, x))$ is computable.

Proof. Denote $\neg R := \mathbb{N}^k \setminus R$. Then:

$$f(\vec{a}) = \mu_x(R(\vec{a}, x)) = \mu_x(\mathbb{1}_R(\vec{a}, x) = 1) = \mu_x(\mathbb{1}_{\neg R}(\vec{a}, x) = 0).$$ $\boxed{e}$

(f) Computable functions are closed under definitions by cases, i.e. if $f_1, \dots, f_m: \mathbb{N}^k \to \mathbb{N}$ are computable functions and $R_1, R_2, \dots, R_m \subseteq \mathbb{N}^k$ are computable relations that form a partition of $\mathbb{N}^k$, then the function $f: \mathbb{N}^k \to \mathbb{N}$ defined by

$$\vec{a} \mapsto \begin{cases} f_1(\vec{a}) & \text{if } R_1(\vec{a}) \text{ holds} \\ f_2(\vec{a}) & \text{if } R_2(\vec{a}) \text{ holds} \\ \vdots & \\ f_m(\vec{a}) & \text{if } R_m(\vec{a}) \text{ holds} \end{cases}$$

is computable.

Proof. Left as HW. $\boxed{\text{f}}$